

Effective field theory for deformed atomic nuclei

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Abstract. We present an effective field theory (EFT) for a model-independent description of deformed atomic nuclei. In leading order this approach recovers the well-known results from the collective model by Bohr and Mottelson. When higher-order corrections are computed, the EFT accounts for finer details such as the variation of the moment of inertia with the band head and the small magnitudes of interband $E2$ transitions. For rotational bands with a finite spin of the band head, the EFT is equivalent to the theory of a charged particle on the sphere subject to a magnetic monopole field.

Submitted to: *Institute of Physics Publishing*
Phys. Scr.

1. Introduction

The ground-breaking papers by Bohr [1] and by Bohr and Mottelson [2] have laid the foundation for our understanding of collective motion in heavy nuclei [3]. In that approach, collective excitations are quantized surface oscillations of a liquid drop, with quadrupole modes dominating at low energies. The five degrees of freedom associated with quadrupole deformations of a spherical surface are chosen as the three Euler angles (that describe the orientation of the nonspherical liquid drop) and two shape parameters β and γ (that describe the amplitudes of the deformation in the body-fixed, i.e. the co-rotating, coordinate system of the liquid drop). The parameter β describes the deformation of an axially symmetric drop, while γ describes deformations that break the axial symmetry.

The Bohr Hamiltonian governs the dynamics of the collective degrees of freedom. The kinetic term is that of a five-dimensional harmonic quadrupole oscillator. Because of rotational invariance, the potential term is a function of the two shape parameters β and γ alone. The recent compendium [4] lists and describes analytical solutions known for special cases. For general potentials, the Bohr Hamiltonian must be solved numerically. Powerful numerical methods have been developed for this purpose only recently [5].

Low-lying spectra of deformed nuclei are characterized by three energy scales. The lowest excitations are quantized rotations. In deformed rare-earth nuclei (in actinides) the associated energy scale ξ is $\xi \approx 80$ keV ($\xi \approx 40$ keV, respectively). The spectra consist of rotational bands built on top of vibrational band heads; the latter are excitations of the β and γ degrees of freedom. The energy scale Ω associated with vibrational excitations is $\Omega \approx 800$ keV ($\Omega \approx 500$ keV, respectively). The low-energy description of even-even deformed nuclei in terms of bosonic collective modes breaks down at an energy scale Λ where single-particle effects that reveal the fermionic nature of the nucleonic degrees of freedom become important. In even-even heavy nuclei, pair breaking occurs at $\Lambda \approx 2\text{--}3$ MeV.

The scales ξ, Ω, Λ are separated. The inequality $\xi \ll \Omega$ holds very well, while $\Omega \ll \Lambda$ holds marginally. These facts suggest that a model-independent description of deformed nuclei in terms of an effective field theory (EFT) should be useful. An EFT has several advantages over phenomenological models. First, the separation of scales allows for the introduction of the small expansion parameters ξ/Λ and Ω/Λ . Power counting can be used to order contributions to the Hamiltonian. Keeping terms up to a given order yields a Hamiltonian with quantifiable theoretical uncertainties. The Hamiltonian can be improved and the errors reduced by including terms of the next order. Second, the Hamiltonian and the interaction currents are treated on an equal footing. Transition operators and observables other than the energy can be treated consistently up to the same order as the Hamiltonian. Third, an EFT makes it possible to establish relationships between observables (instead of relationships between model parameters).

We stress that an EFT approach to nuclear deformation is not of purely theoretical interest only. The Bohr Hamiltonian and other, more general collective models describe the gross properties of even-even deformed nuclei very well. Examples are the rotational spectra on top of vibrations and the strong intra-band $E2$ transitions. Finer details are not reproduced correctly. For instance, in the collective models the moments of inertia typically tend to decrease with increasing vibrational excitation energies, in contrast to the data. The magnitudes of the (weak) inter-band $E2$

transitions are overpredicted by factors 2–10 [6].

In the present article we review recent uses of EFT for the description of deformed nuclei. Some of these [7, 8, 9] have employed “effective theory” and were aimed at a direct comparison with the data. The shortcomings of the models mentioned in the previous paragraph have been addressed successfully [8, 9]. Other papers [10, 11] developed an EFT for “emergent symmetry breaking” (a general phenomenon in finite systems), with special application to deformed nuclei. Here we aim at a unified presentation of these approaches. We will not repeat the detailed derivations given in the above-mentioned articles. We aim at presenting the main physical ideas while keeping the formalism at a minimum.

The use of EFT in nuclear physics is of course not restricted to deformed rotating nuclei. On the contrary, in the last two decades EFTs have been widely used. Chiral EFT exploits the separation of scale between the long-ranged pion exchange (resulting from the spontaneous breaking of chiral symmetry in low-energy quantum chromodynamics) and short-ranged interactions for the systematic construction of the inter-nucleon potentials [12, 13, 14]. Interactions and currents from chiral EFT serve as input for nuclear-structure calculations for nuclei with up to medium mass [15, 16, 17] and neutron-rich matter [18]. Pion-less EFT has been used to describe few-body nuclear systems [19, 20, 21, 22]. Here, the unknown short-ranged interaction is systematically parameterized by contact terms and derivatives. Halo EFT [23, 24, 25] exploits the separation of scale between weakly-bound halo states and higher-lying excitations and describes halo nuclei in terms of valence nucleons that are weakly bound to inert core states.

The article is organized as follows. We introduce emergent symmetry breaking in Sect. 2. In Sect. 3 we review the construction of the EFT for deformed nuclei. Section 4 presents results of this approach, compares the EFT to data and to effective theories, and mentions some interesting problems that have been addressed this way. We give a summary and brief outlook in Sect. 5. Some technical details are presented in Appendix A.

2. Spontaneous versus emergent symmetry breaking

We speak of spontaneous symmetry breaking (both in relativistic and in infinitely extended nonrelativistic systems) when the ground state of a quantum system does not possess the full symmetry of the Hamiltonian. A case in point is the infinite ferromagnet. In the ground state, all spins point in the same direction, violating the rotational invariance of the Hamiltonian. The ground states of two infinite ferromagnets that differ in the orientations of their spins have zero overlap. Indeed, that overlap is the product of the overlaps of infinitely many individual spin states all of which are less than unity. Therefore, a unitary transformation that would link the two ground states, does not exist: The two Hilbert spaces built upon the two ground states are unitarily inequivalent. A rotation is quantum-mechanically described by a unitary transformation in Hilbert space. Therefore, the infinite ferromagnet cannot rotate. EFT offers a way to determine the low-lying excitations of such systems using symmetry arguments only. In the ferromagnet, these are spin waves with arbitrarily long wave lengths. The EFT construction uses the Goldstone theorem. Low-lying excitations are given in terms of the Nambu-Goldstone modes.

Spontaneous symmetry breaking does not occur in finite systems. Ground states of finite ferromagnets, for instance, that differ in spin orientation have nonzero

overlap and are linked by unitary transformations describing rotations. Therefore, a finite ferromagnet can rotate about any axis perpendicular to the symmetry axis of the ground state, and rotational invariance is restored. However, there is a smooth transition to the case of spontaneously broken symmetry: As the size of the ferromagnet increases, the overlap of ground states oriented in different directions decreases exponentially and becomes zero as the size tends to infinity. Then rotational motion cannot occur. We speak of emergent symmetry breaking in a finite system when spontaneous symmetry breaking occurs in the limit of infinite system size. Thus, emergent symmetry breaking is the precursor of spontaneous symmetry breaking. The ground state of the finite system possesses the full symmetry of the Hamiltonian. Nevertheless, some of the low-lying excitations are akin to those of the infinite system. The EFT construction is, thus, expected to apply also for emergent symmetry breaking, augmented, of course, by degrees of freedom that account for rotational motion.

That is the idea we pursue in the present paper. Nuclei are invariant under rotations, and angular momentum is a good quantum number. But heavy deformed nuclei possess an intrinsic ground state that is deformed and, thus, breaks the rotational invariance of the Hamiltonian in the limit of infinite system size. We consider that a case of emergent symmetry breaking [26]. We modify the EFT approach to infinite systems so as to take account of rotational motion. The approach is expected to work best for large systems (with large moments of inertia) as we are then close to the infinite case. The energy scales discussed in the introduction show that that condition is met very well in deformed nuclei. The broken symmetry is rotational $SO(3)$, the deformed ground state is invariant under $SO(2)$ rotations about the body-fixed symmetry axis. Hence, our generalized EFT is based on Nambu-Goldstone modes constructed in the $SO(3)/SO(2)$ coset space, a nonlinear realization of the underlying symmetry.

We compare that approach to the standard description of nuclei. Practically all theoretical approaches to nuclei employ spherical tensors as fundamental tools, i.e., use the Wigner-Weyl (linear) realization of $SO(3)$ symmetry. The Bohr Hamiltonian and the general collective models, for instance, employ quadrupole degrees of freedom, i.e., spherical tensors of rank two, while the interacting boson model (IBM) [27] employs in addition to the d bosons also an s boson. In Bohr's original approach [1], the transformation from quadrupole degrees of freedom to Euler angles and shape parameters is from the laboratory to the body-fixed system. That non-linear transformation is complicated, and it is not obvious how to construct, for instance, higher-order kinetic terms directly in the body-fixed coordinates. Moreover, in the geometric and algebraic collective models, the description of deformed nuclei requires large basis sets when the basis of the quadrupole oscillator is used. Efficient basis construction schemes and formulas for many relevant matrix elements have only been given recently [5, 28].

In contrast to these bosonic descriptions, more microscopic approaches such as the nuclear shell model are based on fermionic degrees of freedom, i.e., on spherical tensors of half-integer rank. For deformed nuclei, such microscopic approaches are complicated and/or numerically expensive. Mean-field calculations, for instance, yield a product state that breaks rotational invariance, and projection methods need be employed for the restoration of rotational symmetry [29, 30]. Likewise, the Nilsson model [31] constructs deformed states in an intrinsic (co-rotating) frame, and one is left with the problem to couple the intrinsic states to rotations. *Ab initio* methods

have recently succeeded in describing emergent phenomena such as rotational bands in light nuclei [32, 33]. These tour-de-force computations employ large model spaces [16] and/or complicated basis states [34] for the description of phenomena linked to rotational motion.

3. EFT for deformed nuclei

Effective field theories are based on the symmetries and the pattern of symmetry breaking relevant for the physical system under consideration. For systems with spontaneous symmetry breaking, Weinberg [35], Coleman *et al.* [36], and Callan *et al.* [37] devised methods for the construction of effective Lagrangeans. These approaches are reviewed in Refs. [38, 39, 40]. For deformed nuclei, we deal with emergent symmetry breaking from rotational $SO(3)$ to axial $SO(2)$ symmetry. The EFT approach to these finite systems is to a large extent similar to systems such as (anti-)ferromagnets [38, 41, 42, 43, 44] that also exhibit a spontaneous breaking of $SO(3)$ to $SO(2)$ symmetry, but modifications arise due to the emergent character of the symmetry breaking.

In systems with a spontaneous breaking of the symmetry group \mathcal{G} to a subgroup \mathcal{H} , the low-energy degrees of freedom are Nambu-Goldstone (NG) bosons which parameterize the coset space \mathcal{G}/\mathcal{H} [36, 37]. In our case, $\mathcal{G} = SO(3)$ and $\mathcal{H} = SO(2)$. The NG bosons are fields that depend on spatial and temporal coordinates. In the infinite system, a purely time-dependent mode (that is constant in space) is excluded because such a mode connects unitarily inequivalent Hilbert spaces (in the ferromagnet: symmetry axes that point in macroscopically different directions). In finite systems with emergent symmetry breaking, that zero mode cannot be neglected. In our case, the mode corresponds to overall rotations of the system and connects Hilbert spaces that become inequivalent as the system size approaches infinity. It is that mode that restores the rotational symmetry and allows us to label states by total angular momentum. Such zero modes are also important for finite-size effects in numerical simulations of field theories in a finite volume [45, 46, 47]. We summarize the construction of an EFT based on emergent symmetry breaking for the case of deformed nuclei developed in Refs. [10, 11].

For deformed nuclei, the relevant low-energy degrees of freedom are NG fields $\psi_x(\mathbf{x}, t)$ and $\psi_y(\mathbf{x}, t)$, and rotations are described by angles $\phi(t), \theta(t)$. Here, $\mathbf{x} = (x_1, x_2, x_3)$ denotes the three spatial coordinates of the fields. The NG fields and the rotation angles parameterize the coset space $SO(3)/SO(2)$ which is isomorphic to the two-sphere S^2 . We consider

$$U \equiv g(\phi, \theta) u(\psi_x, \psi_y) , \quad (1)$$

$$g(\phi, \theta) \equiv e^{-i\phi\hat{J}_z} e^{-i\theta\hat{J}_y} , \quad (2)$$

$$u(\psi_x, \psi_y) = e^{-i\psi_x\hat{J}_x - i\psi_y\hat{J}_y} . \quad (3)$$

The three angular-momentum operators \hat{J}_k , $k = x, y, z$ generate rotations around the k -axes in the space-fixed coordinate system. They fulfill the usual commutation relations $[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$ (and cyclic permutations) \ddagger . The physical picture is this. We choose the z -axis of the laboratory system such that it initially coincides with the

\ddagger We could also use operators that generate rotations around the body-fixed k -axes. Then the commutation relations carry different signs. That would not change the physical picture.

symmetry axis of the deformed nucleus, and we apply $U = gu$ of Eq. (1). For small $\psi_x(\mathbf{x}, t)$ and $\psi_y(\mathbf{x}, t)$ the operator u causes position-dependent small-angle distortions of each volume element (each nucleon) of the liquid drop (of the nucleus consisting of nucleons, respectively). These distortions have to be quantized. The NG fields $\psi_x(\mathbf{x}, t)$ and $\psi_y(\mathbf{x}, t)$, thus, induce quantized vibrations at the energy scale Ω . The operator u is followed by g which generates an overall rotation of the vibrating nucleus. The new symmetry axis is oriented along the vector

$$\vec{e}_r \equiv \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}. \quad (4)$$

The vector \vec{e}_r is defined in actual physical space. The vector \mathbf{x} refers to the coordinates of the constituents of the nucleus. For that reason the two vectors are denoted differently. We are going to use a small-amplitude approximation for the NG fields $\psi_x(\mathbf{x}, t)$ and $\psi_y(\mathbf{x}, t)$ whereas the rotation described by the angles θ and ϕ is fully taken into account.

3.1. Transformation properties under rotations

To study the transformation properties of our degrees of freedom under rotations, we act onto U defined in Eq. (1) with a general rotation $f(\alpha, \beta, \gamma) = \exp(-i\alpha\hat{J}_z) \exp(-i\beta\hat{J}_y) \exp(-i\gamma\hat{J}_z)$ with Euler angles (α, β, γ) . From Eq. (2) we have

$$f(\alpha, \beta, \gamma) = g(\alpha, \beta)h(\gamma). \quad (5)$$

Here

$$h(\gamma) = e^{-i\gamma\hat{J}_z} \quad (6)$$

is a pure rotation around the z -axis. Any rotation f acting on a rotation g yields a new rotation f' . Thus,

$$\begin{aligned} f(\alpha, \beta, \gamma)g(\phi, \theta) &= f(\alpha', \beta', \gamma') \\ &= g(\phi', \theta')h(\gamma'). \end{aligned} \quad (7)$$

Eq. (7) defines the rotated angles $(\phi', \theta', \gamma')$. Here $\gamma' = \gamma'(\alpha, \beta, \gamma, \phi, \theta)$ is, in general, a complicated function of its arguments. Acting with f onto U of Eq. (1) yields

$$\begin{aligned} f(\alpha, \beta, \gamma)U &= g(\phi', \theta')h(\gamma')u(\psi_x, \psi_y) \\ &= g(\phi', \theta') [h(\gamma')u(\psi_x, \psi_y)h^{-1}(\gamma')] h(\gamma') \\ &= g(\phi', \theta')u(\psi'_x, \psi'_y)h(\gamma'). \end{aligned} \quad (8)$$

The last line defines the rotated NG fields ψ'_x and ψ'_y through

$$u(\psi'_x, \psi'_y) \equiv h(\gamma')u(\psi_x, \psi_y)h^{-1}(\gamma'). \quad (9)$$

The nontrivial result of these transformation properties is that the NG fields transform linearly (albeit with a complicated transformation angle γ'), while the rotation angles (ϕ, θ) transform nonlinearly. Equation (4) shows that the rotation angles (ϕ, θ) can be viewed as azimuthal and polar angles that parameterize the two-sphere; under rotations, they transform accordingly. The NG fields can be viewed as parameterizing the tangent plane of the two-sphere at (ϕ, θ) ; under rotations they transform as the components of a vector in the tangent plane, i.e., they undergo an $SO(2)$ rotation about the angle γ' around an axis perpendicular to the tangent plane.

In our parameterization the limit to the infinite system is not smooth. In the infinite system (ϕ, θ) assume constant (i.e., time-independent) values, and we can for simplicity set $g = 1$. Then under rotations $fU = fgu = fu$, and the NG fields (ψ_x, ψ_y) transform non-linearly. Details about the transformation properties of the EFT are presented in the Appendix A.

3.2. Effective Lagrangean

The Lagrangean must be invariant under rotations. It is obtained as a linear combination of invariants that are formed from the NG fields and their derivatives with respect to time and space. The coefficients of the linear combination are the parameters of the theory. These must be adjusted to the data. The full construction, extensively discussed in Refs. [7, 8, 10, 9, 11], shows that invariants involving neither spatial nor temporal derivatives do not occur. The invariants are ordered using the energy scales introduced in Section 2. The resulting power-counting scheme is similar to the one used in Section 3.3.2 below for the Hamiltonian and is not discussed here, see Refs. [10, 11]. The resulting Lagrangean (containing, for instance, only leading-order or leading-plus-next-to-leading-order terms) defines a classical field theory. It has (at least) two constants of motion, the energy and the total angular momentum of the system.

In the present Section we do not follow the procedure of Refs. [10, 11] for the construction of the invariants but employ a shortcut suggested by the geometric picture given above. As the coset space $SO(3)/SO(2)$ is isomorphic to the two-sphere, the dynamics of rotations is that of a particle on that sphere. The rotation angles θ and ϕ appear as arguments of the radial vector

$$\vec{r} = r\vec{e}_r \quad (10)$$

that parameterizes the surface of the two-sphere sphere of radius r . The parameter r is a low-energy constant (LEC) to be determined later. The velocity of the particle is

$$\vec{v} \equiv r\partial_t\vec{e}_r = r\dot{\theta}\vec{e}_\theta + r\dot{\phi}\sin\theta\vec{e}_\phi. \quad (11)$$

Here, the dot over a variable denotes its time derivative. The vector \vec{v} lies in the plane tangent to the sphere at the point (ϕ, θ) . The plane is spanned by the azimuthal and polar unit vectors

$$\vec{e}_\theta \equiv \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix} \text{ and } \vec{e}_\phi \equiv \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \quad (12)$$

Under rotations, the components $(v_\theta, v_\phi) = (r\dot{\theta}, r\dot{\phi}\sin\theta)$ of the velocity vector transform in the tangential plane as do the NG fields (ψ_x, ψ_y) .

For the time derivatives of the NG fields we use the covariant derivative

$$D_t = \partial_t - i\dot{\phi}\cos\theta\hat{J}_z. \quad (13)$$

The operator D_t projects the time derivative of a vector in the tangential plane back onto the tangential plane. When D_t acts on the NG fields (ψ_x, ψ_y) , the operator \hat{J}_z in Eq. (13) acts as $-i(\psi_x\partial\psi_y - \psi_y\partial\psi_x)$. That yields

$$D_t \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \dot{\psi}_x \\ \dot{\psi}_y \end{pmatrix} + \dot{\phi}\cos\theta \begin{pmatrix} \psi_y \\ -\psi_x \end{pmatrix}. \quad (14)$$

Spatial derivatives of the NG fields are written as $\partial_k\psi_x(\mathbf{x}, t)$ or $\partial_k\psi_y(\mathbf{x}, t)$ with $k = 1, 2, 3$. Here $\partial_k \equiv \partial/\partial x_k$ denotes the partial derivative with respect to the

component x_k of the coordinate vector $\mathbf{x} = (x_1, x_2, x_3)$. As is the case for the NG fields themselves, for each point \mathbf{x} the derivative fields lie in the plane tangent to the two-sphere at (ϕ, θ) and under rotations transform accordingly. For a coordinate system chosen such that the symmetry axis of the deformed nucleus is initially along $\mathbf{x} = (0, 0, x_3)$, combinations of derivatives that respect that symmetry such as $\partial_1^2 + \partial_2^2$ or ∂_3^2 are permitted.

In summary, scalars constructed from the vectors $(\dot{\theta}, \dot{\phi} \sin \theta)$, $(D_t \psi_x, D_t \psi_y)$, and $\partial_k(\psi_x, \psi_y)$ (that all lie within the tangential plane) that are invariant under $SO(2)$ rotations are formally invariant under $SO(3)$ rotations and form the building blocks of the effective Lagrangean. Invariants such as $(\psi_x^2 + \psi_y^2)$ that involve only the NG vectors (ψ_x, ψ_y) and not their derivatives are not allowed. However, invariants of that type may be used as factors multiplying any of the scalars formed from the derivatives.

As for discrete symmetries, we consider parity, \mathcal{R} parity, and time-reversal. The fields ψ_x and ψ_y have positive parity because the corresponding generators \hat{J}_x and \hat{J}_y do [see Eq. (1)]. Thus, all states that result from the EFT construction have positive parity. \mathcal{R} parity is defined as the result of a rotation by π around any axis perpendicular to the axis of axial symmetry. In nuclei that operation maps the ground state onto itself, so that nuclei have positive \mathcal{R} parity [3]. That fact distinguishes nuclei from molecules. Time-reversal invariance restricts invariants to terms that contain an even number of time derivatives.

The effective Lagrangean obtained that way is rotationally invariant. That fact implies that the total angular momentum \vec{I} is conserved. It is given by (see Appendix A for details)

$$\vec{I} = p_\theta \vec{e}_\phi - \frac{p_\phi - K \cos \theta}{\sin \theta} \vec{e}_\theta + K \vec{e}_r. \quad (15)$$

Here p_ϕ and p_θ are the canonical momenta of ϕ and θ , respectively, defined as usual in terms of partial derivatives of the Lagrangean. The contribution to the angular momentum of the NG fields is

$$K \equiv \int d^3x (\psi_x p_y - \psi_y p_x), \quad (16)$$

where p_x and p_y denote the canonical momenta of the NG fields ψ_x and ψ_y , respectively. For every point \mathbf{x} of the fields, the term $(\psi_x p_y - \psi_y p_x)$ is a scalar under rotations. Hence K is a scalar as well. The square of the total angular momentum is

$$I^2 = p_\theta^2 + \frac{1}{\sin^2 \theta} (p_\phi^2 - 2K p_\phi \cos \theta + K^2). \quad (17)$$

Eqs. (15) and (16) follow from an application of the Noether theorem to the effective Lagrangean, see Refs. [10, 11].

3.3. Effective Classical Hamiltonian

In Refs. [10, 11], a Legendre transformation of the effective Lagrangean leads to the effective Hamiltonian, which in turn is quantized and employed for the computation of spectra and transitions. That procedure is straightforward but tedious, especially when terms beyond leading order are considered. In the present Section we show how an effective Hamiltonian that is invariant under rotations may be constructed directly.

3.3.1. Kinetic Terms. The Legendre transformation effectively replaces time derivatives of variables by their canonical momenta. Spatial derivatives $\partial_k(\psi_x, \psi_y)$ are not affected by the transformation. Therefore, we first address the kinetic terms in the effective Hamiltonian.

The vector

$$\vec{p} = \frac{p_\theta}{r} \vec{e}_\theta + \frac{p_\phi}{r \sin \theta} \vec{e}_\phi \quad (18)$$

denotes the momentum of a particle on the two-sphere with radius r . It is conjugate to the vector \vec{r} in Eq. (10). The vector \vec{p} lies in the tangent plane of the two-sphere at the point (θ, ϕ) . The components p_θ/r and $p_\phi/(r \sin \theta)$ transform accordingly. The momenta of the NG fields (p_x, p_y) are likewise components of a vector in the tangent plane and under rotations transform accordingly as well. Kinetic terms in the Hamiltonian that are invariant under rotations are, thus, scalars in the tangential plane.

Rotations and surface vibrations are coupled in a subtle way. That is seen by rewriting the angular momentum in Eq. (15) as

$$\vec{I} = \vec{r} \times \left(\vec{p} - \frac{K \cot \theta}{r} \vec{e}_\phi \right) + K \vec{e}_r. \quad (19)$$

The term $K \vec{e}_r$ is the intrinsic angular momentum along the symmetry axis of the deformed nucleus. The cross product in Eq. (19) is the angular momentum associated with the rotor (i.e. with a particle on the sphere of radius r). The vector

$$\vec{p} - \frac{K \cot \theta}{r} \vec{e}_\phi. \quad (20)$$

may be viewed as the “gauged” momentum of a particle on the sphere subject to a vector potential

$$\vec{A} \equiv \frac{K \cot \theta}{r} \vec{e}_\phi \quad (21)$$

with associated magnetic field

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \\ &= \left(\vec{e}_r \partial_r + \vec{e}_\theta \frac{1}{r} \partial_\theta + \vec{e}_\phi \frac{1}{r \sin \theta} \partial_\phi \right) \times \vec{A} \\ &= -\frac{K}{r^2} \vec{e}_r. \end{aligned} \quad (22)$$

The field \vec{B} is normal to the surface of the sphere. It is obviously invariant under rotations. We note that

$$\frac{1}{2} \left(\vec{p} - \frac{K \cot \theta}{r} \vec{e}_\phi \right)^2 = \frac{\vec{I}^2 - K^2}{2r^2} \quad (23)$$

is the Hamiltonian of an axially symmetric rotor, with r^2 the moment of inertia. We also note that rotations change the vector potential (21) by a gradient of a scalar function. Thus, rotations induce an unobservable gauge transformation but otherwise leave gauged kinetic terms invariant. Details are presented in Appendix A.

Vibrational and rotational degrees of freedom are, thus, coupled by the gauge field \vec{A} . The vector potential (21) is that of a magnetic monopole with charge K at the center of the sphere [48, 49]. (For a monopole it would be required that the “charge” K is integer or half integer.) Alternatively we may say that the intrinsic

angular momentum $K\vec{e}_r$ generates a magnetic moment. The resulting magnetic field at the point \vec{r} (which points in the radial direction) couples vibrational and rotational motion.

This insight allows one develop an EFT for rotational bands with a finite spin S of the band head, based on rotational degrees of freedom (θ, ϕ) alone (i.e. without introduction of vibrational degrees of freedom). We note that odd-mass nuclei and odd-odd nuclei have finite spins in their ground states. The Hamiltonian

$$\frac{1}{2} \left(\vec{p} - \frac{S \cot \theta}{r} \vec{e}_\phi \right)^2 = \frac{\vec{I}^2 - S^2}{2r^2} \quad (24)$$

with S being an integer or a half-integer constant is clearly invariant under rotations. For $|S| > 1/2$, this Hamiltonian indeed yields the leading-order description of nuclei with a ground-state spin S [3]. Two comments are in order. Let us first consider time-reversal invariance. Under time reversal $\vec{p} \rightarrow -\vec{p}$, and $K \rightarrow -K$, but S is a constant. This confirms that the Hamiltonian (23) is invariant under time reversal (it was constructed this way) in contrast to the Hamiltonian (24). This explains why the latter could not directly be derived within the EFT presented in this paper. Second, Coriolis forces modify the spectrum of the Hamiltonian (24) at leading order for spins $S = 1/2$. We remind the reader that Coriolis forces couple the angular momentum of the rotor to the nuclear spin S , and they can induce spin flips. Such spin flips correspond to $S \rightarrow -S$ and are only possible if S is a dynamical degree of freedom (and not a constant). Thus, the Hamiltonian (24) describes the leading-order physics only for $|S| > 1/2$, because only then are spin-flips higher energetic excitations and beyond leading order. For a full-fledged EFT of deformed odd-mass nuclei, one would need to couple nucleons to the rotor.

3.3.2. Power counting. Power counting is based on the scales ξ and Ω defined in Section 2 and on the relations

$$\begin{aligned} \phi, \theta &\sim \mathcal{O}(1) , \\ \dot{\phi}, \dot{\theta} &\sim \xi , \\ p_\phi, p_\theta &\sim \mathcal{O}(1) , \\ \psi_x, \psi_y &\sim \sqrt{\xi/\Omega} \ll 1 , \\ \dot{\psi}_x, \dot{\psi}_y &\sim \sqrt{\xi\Omega} , \\ p_x, p_y &\sim \sqrt{\Omega/\xi} . \end{aligned} \quad (25)$$

The first three relations reflect the kinematics of rotational motion. The angles ϕ and θ range from zero to 2π and are, thus, of order unity. The right-hand side of Eq. (23) shows that r^2 is the moment of inertia. By definition, the rotational Hamiltonian is of order ξ . Thus, $r^2 \sim \xi^{-1}$. Writing the rotational Hamiltonian in terms of either $\dot{\phi}$ and $\dot{\theta}$ or of p_ϕ and p_θ one finds the second and the third of relations (25). The energy of the vibrational modes is of order Ω and so are, therefore, the ratios $\dot{\psi}_x/\psi_x$ and $\dot{\psi}_y/\psi_y$. The occurrence of ξ in the factors $\sqrt{\xi/\Omega}$ and $\sqrt{\xi\Omega}$ is caused by the term that couples rotational and vibrational motion. Relations three and six show that momenta scale inversely to their corresponding coordinates. The inequality in relation four expresses the assumption stated in Section 1. That assumption implies that the amplitudes of the vibrational modes are small.

3.3.3. *Effective Hamiltonian to order ξ* The leading terms describe vibrations and are of order Ω . In that order we have

$$H_\Omega = \int d^3x \left\{ \frac{p_x^2 + p_y^2}{2M} + \frac{M}{2} \left[\omega_\parallel^2 ((\partial_3 \psi_x)^2 + (\partial_3 \psi_y)^2) + \omega_\perp^2 ((\partial_1 \psi_x)^2 + (\partial_2 \psi_x)^2 + (\partial_1 \psi_y)^2 + (\partial_2 \psi_y)^2) \right] \right\}. \quad (26)$$

The low-energy constants M , ω_\parallel , and ω_\perp scale as Ω^{-1} , Ω , and Ω , respectively. We note that terms involving spatial derivatives of the NG fields play the role of potential terms. Integration by parts yields

$$H_\Omega = \int d^3x \left\{ \frac{p_x^2 + p_y^2}{2M} - \frac{M}{2} \left[\omega_\parallel^2 (\psi_x \partial_3^2 \psi_x + \psi_y \partial_3^2 \psi_y) + \omega_\perp^2 (\psi_x (\partial_1^2 + \partial_2^2) \psi_x + \psi_y (\partial_1^2 + \partial_2^2) \psi_y) \right] \right\}. \quad (27)$$

We decompose the NG fields

$$\psi_x(\mathbf{x}, t) = \sum_\alpha x_\alpha(t) \chi_\alpha(\mathbf{x}), \quad \psi_y(\mathbf{x}, t) = \sum_\alpha y_\alpha(t) \chi_\alpha(\mathbf{x}) \quad (28)$$

into orthonormalized eigenmodes of the Helmholtz equation

$$-\left[\omega_\parallel^2 \partial_3^2 + \omega_\perp^2 (\partial_1^2 + \partial_2^2) \right] \chi_\alpha(\mathbf{x}) = \omega_\alpha^2 \chi_\alpha(\mathbf{x}). \quad (29)$$

Axial symmetry implies that many eigenfunctions are pairwise degenerate in energy. We assume that the spectrum $\omega_0 < \omega_1 \leq \omega_2 \leq \dots$ is ordered, and that the lowest eigenvalue is not degenerate. The momenta are decomposed correspondingly,

$$p_x(\mathbf{x}, t) = \sum_\alpha p_{x;\alpha}(t) \chi_\alpha(\mathbf{x}), \quad p_y(\mathbf{x}, t) = \sum_\alpha p_{y;\alpha}(t) \chi_\alpha(\mathbf{x}). \quad (30)$$

The components $p_{x;\alpha}$ and $p_{y;\alpha}$ are canonical momenta of the components x_α and y_α , respectively. The Hamiltonian (27) becomes

$$H_\Omega = \sum_\alpha \left[\frac{p_{x;\alpha}^2 + p_{y;\alpha}^2}{2M} + \frac{M}{2} \omega_\alpha^2 (x_\alpha^2 + y_\alpha^2) \right]. \quad (31)$$

Calculation of the ω_α would require a specific model for the shape of the deformed nucleus and for the boundary conditions imposed at the surface. That is of no interest here. In the EFT, only the lowest-energy parameters ω_α enter and need to be adjusted to data.

The total Hamiltonian down to order ξ is obtained by adding to H_Ω the kinetic terms constructed in Section 3.3.1. That gives

$$H_\xi = H_\Omega + \frac{I^2 - K^2}{2r^2}. \quad (32)$$

Vibrational and rotational motion are coupled via the gauge term in Eq. (23).

For the construction of invariants at higher order, we note that it might be useful to consider the vectors

$$\begin{aligned} \vec{r}_\alpha &\equiv x_\alpha \vec{e}_\theta + y_\alpha \vec{e}_\phi \\ \vec{p}_\alpha &\equiv p_{x;\alpha} \vec{e}_\theta + p_{y;\alpha} \vec{e}_\phi \end{aligned} \quad (33)$$

that are in the tangent plane. As an example, we note that the intrinsic angular momentum (16) can be written as

$$K \vec{e}_r = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha. \quad (34)$$

3.4. Quantization

For the vibrational momenta, the quantization rules are standard,

$$\begin{aligned} p_{x;\alpha} &= -i\partial_{x_\alpha} , \\ p_{y;\alpha} &= -i\partial_{y_\alpha} , \end{aligned} \quad (35)$$

and the resulting spectrum is that of infinitely many uncoupled two-dimensional harmonic oscillators with an $SO(2)$ symmetry and frequencies ω_α . The spectrum of each $SO(2)$ symmetric oscillator is

$$E_\alpha(n_\alpha, k_\alpha) = \omega_\alpha(2n_\alpha + |k_\alpha|) . \quad (36)$$

Here n_α with $n_\alpha = 0, 1, 2, \dots$ is the principal quantum number and k_α with $k_\alpha = 0, \pm 1, \pm 2, \dots$ is the projection of the angular momentum of the vibrational modes. We neglect zero-point energies. We have

$$K = \sum_\alpha (x_\alpha p_{y;\alpha} - y_\alpha p_{x;\alpha}) . \quad (37)$$

The eigenvalue k of the operator K depends upon the eigenstate of the Hamiltonian that K is acting on and is given by

$$k = \sum_\alpha k_\alpha . \quad (38)$$

The quantization of angular momentum is standard. In Eq. (23) we replace the operator K by its eigenvalue k . The eigenfunctions of the rotational Hamiltonian obtained by that replacement are Wigner D functions [50],

$$\frac{\vec{I}^2 - k^2}{2r^2} D_{mk}^I(\phi, \theta, 0) = \frac{I(I+1) - k^2}{2r^2} D_{mk}^I(\phi, \theta, 0) , \quad (39)$$

with $I = |k|, |k| + 1, |k| + 2, \dots$. The moment of inertia is r^2 . Due to \mathcal{R} parity, only linear combinations of wave functions $D_{mk}^I + (-1)^k D_{m-k}^I$ are admissible. The spectrum consists of rotational bands on top of vibrational band heads with integer spin $|k|$.

3.5. Higher-order terms

The amplitudes of the surface vibrations are small and of order $\varepsilon \sim \Omega/\Lambda$. Terms of next order contain higher powers of these amplitudes and are, thus, of order $\Omega\varepsilon$. Relevant kinetic invariants are then

$$\begin{aligned} & \int d^3x (\psi_x p_y - \psi_y p_x)^2 , \\ & \int d^3x (\psi_x^2 + \psi_y^2)(p_x^2 + p_y^2) , \\ & \frac{\cos^2 \theta}{\sin^2 \theta} \int d^3x (\psi_x p_y - \psi_y p_x)^2 . \end{aligned} \quad (40)$$

The expansion coefficients of p_x, p_y are quantized as in Eqs. (35). The first two terms are constructed from invariants in a very obvious way. The last term results from the coupling of rotational motion and vibrations. As shown by the last term in the covariant derivative in Eq. (13), that coupling is proportional to $\cos \theta$ and yields the factor $\cos^2 \theta$. The expression (11) for the velocity shows that replacing $\dot{\phi}$ by

the conjugate momentum p_ϕ produces a factor $(\sin \theta)^{-1}$, hence the factor $(\sin \theta)^{-2}$. Explicitly we may use Eq. (19) to write $\int d^3x \vec{I}^2$ as

$$\int d^3x \left[\vec{r} \times \left(\vec{p} - \frac{(\psi_x p_y - \psi_y p_x) \cot \theta}{r} \vec{e}_\phi \right) + (\psi_x p_y - \psi_y p_x) \vec{e}_r \right]^2. \quad (41)$$

The integration extends only over the arguments of ψ_x, ψ_y, p_x, p_y . The square of the term proportional to \vec{e}_ϕ yields the third expression (40). An alternative view on the occurrence of the term proportional to $\cot^2 \theta$ is given in the Appendix A.

Expanding the first expression (40) into eigenmodes yields

$$\sum_{\alpha\beta\gamma\delta} \chi_{\alpha\beta\gamma\delta} (x_\alpha p_{y;\beta} - y_\alpha p_{x;\beta}) (x_\gamma p_{y;\delta} - y_\gamma p_{x;\delta}) \quad (42)$$

with

$$\chi_{\alpha\beta\gamma\delta} \equiv \int d^3x \chi_\alpha(\mathbf{x}) \chi_\beta(\mathbf{x}) \chi_\gamma(\mathbf{x}) \chi_\delta(\mathbf{x}). \quad (43)$$

Because of the axial symmetry of the eigenfunctions $\chi_\mu(\mathbf{x})$, the expression $\chi_{\alpha\beta\gamma\delta}$ vanishes unless the azimuthal quantum numbers contained in the labels $(\alpha, \beta, \gamma, \delta)$ sum up to zero. Thus, we can rewrite the first two invariants in Eq. (40) as

$$\begin{aligned} & \sum_{\alpha\beta\gamma\delta} \chi_{\alpha\beta\gamma\delta} (\vec{r}_\alpha \times \vec{p}_\beta) \cdot (\vec{r}_\gamma \times \vec{p}_\delta), \\ & \sum_{\alpha\beta\gamma\delta} \chi_{\alpha\beta\gamma\delta} (\vec{r}_\alpha \cdot \vec{r}_\beta) (\vec{p}_\gamma \cdot \vec{p}_\delta). \end{aligned} \quad (44)$$

Potential terms in the Hamiltonian involve invariants constructed from the spatial derivatives of the fields ψ_x and ψ_y . Examples are

$$\begin{aligned} & \int d^3x (\psi_x \nabla \psi_y - \psi_y \nabla \psi_x)^2, \\ & \int d^3x (x^2 + y^2) ((\nabla x)^2 + (\nabla y)^2). \end{aligned} \quad (45)$$

Here ∇ stands for the gradient with respect to \mathbf{x} and acts onto the modes (28). Formally, such terms can be written as

$$\begin{aligned} & \sum_{\alpha\beta\gamma\delta} \tilde{\chi}_{\alpha\beta\gamma\delta} (\vec{r}_\alpha \times \vec{r}_\beta) \cdot (\vec{r}_\gamma \times \vec{r}_\delta), \\ & \sum_{\alpha\beta\gamma\delta} \tilde{\chi}_{\alpha\beta\gamma\delta} (\vec{r}_\alpha \cdot \vec{r}_\beta) (\vec{r}_\gamma \cdot \vec{r}_\delta). \end{aligned} \quad (46)$$

Here, $\tilde{\chi}_{\alpha\beta\gamma\delta}$ results from an integral similar to Eq. (43) but with derivatives in the integrand.

4. Results

In this Section we review some of the results obtained with the EFT approach to deformed nuclei. First, we discuss the spectrum of the Hamiltonian (32). Second, we briefly discuss the coupling of the EFT degrees of freedom to electromagnetic fields and some electromagnetic transitions. Third, we consider how the EFT constructed in this paper reduces to an effective theory when the number of NG modes is truncated to a (small) set. We end with a brief discussion of common features of and differences between the EFT and collective models.

4.1. Spectra

Our EFT applies equally to low-lying states in axially symmetric nuclei and in axially symmetric molecules. In both cases, the ground state has spin $k = 0$, and the ground-state rotational band has spins $I = 0, 2, 4, \dots$. The results differ for the next states in the spectrum, however, because nuclei possess positive \mathcal{R} parity. That excludes a positive-parity state with $k = 1$. Such a state requires the breaking of a Cooper pair [51] since two identical fermions in a single j shell can only couple to even angular momenta. Although our EFT approach does not include fermionic degrees of freedom, the effects of pairing are seen to be indirectly taken into account. In axially symmetric molecules, the lowest vibrational state has a single quantum in the $\alpha = 0$ mode, quantum numbers $k = k_0 = 1$ [52], excitation energy ω_0 , and negative \mathcal{R} parity. In nuclei, that state is excluded. The lowest-lying vibrational states have quantum numbers $(n_0 = 1, k_0 = 0)$ and $(n_0 = 0, |k_0| = 2)$. For the leading-order Hamiltonian, these are degenerate at energy $2\omega_0$. The first of these ($k = 0$) is commonly referred to as the “ β ” vibration. The second ($k = 2$) is the “ γ ” vibration. The degeneracy is lifted by terms of higher order in the Hamiltonian. In most deformed nuclei there is indeed a quasi-degenerate doublet of excited vibrational bands for which the difference in band-head energy is much smaller than the excitation energy $2\omega_0$. Beyond the ground-state band and the nearly degenerate $k = 0$ and $k = 2$ excited bands, the vibrational spectra of nuclei become non-universal. Details depend on the precise values of the energies $\omega_0 < \omega_1 \leq \omega_2 \leq \dots$, and different nuclei are expected to exhibit different vibrational spectra. Within the leading-order Hamiltonian, all rotational bands have identical rotational constants (or moments of inertia). Differences in rotational constants are caused by terms of higher order in the power counting [8].

It is of interest to compare these theoretical results with the extensive data for ^{168}Er [53] and ^{162}Dy [54]. In ^{168}Er , the head of the $k = 2$ band is at 821 keV, that of the excited $k = 0$ band is at 1217 keV. Thus, $2\omega_0 \approx 1$ MeV. The splitting between the two states is about 40% of $2\omega_0$. The rotational excitation energies are less than $\xi \approx 80$ keV so that $\xi/\Omega \approx 1/10$. The moments of inertia for the ground-state band and for the excited $k = 2$ and $k = 0$ bands are $r^{-2} \approx 27$ keV, $r^{-2} \approx 25$ keV, and $r^{-2} \approx 20$ keV, respectively. The lowest negative-parity state has $k^\pi = 0^-$ and is at about 1100 keV, while the lowest $k^\pi = 1^-$ state is at about 1360 keV.

For ^{162}Dy , the excited $k = 2$ vibrational state is at about 888 keV, the lowest $k = 0$ excited state is at about 1400 keV. Thus $2\omega_0 \approx 1.1$ MeV, and the splitting of the $k = 0$ and $k = 2$ states amounts to almost 50% of $2\omega_0$. The moments of inertia of the ground-state band, the $k = 2$, and the $k = 0$ bands are $r^{-2} \approx 27$ keV, $r^{-2} \approx 25$ keV, and $r^{-2} \approx 18$ keV, respectively.

4.2. Coupling to electromagnetic fields

In this Subsection we fill a gap. In the main part of the paper we have not addressed the coupling of the EFT degrees of freedom to electromagnetic fields. We do so now and work in the Coulomb gauge. Then we only need to consider the vector potential $\vec{A}(\vec{r})$. The gauging of the rotational degrees of freedom (θ, ϕ) is straightforward and has been discussed in detail in Ref. [9]. One finds that expression (18) is changed into

$$\vec{p} \rightarrow \vec{p} - q\vec{A}. \quad (47)$$

Here, the charge q is a LEC and can be adjusted to data using a single transition within the ground-state band.

We turn to the electromagnetic coupling of the NG fields. We recall that for every point \mathbf{x} , the fields $\psi_x(\mathbf{x})$ and $\psi_y(\mathbf{x})$ “live” in the tangential plane of the two-sphere at \vec{r} . We also recall the expansions (28) and (30) for the fields and the associated momenta. The latter, given by

$$\vec{p}_\alpha \equiv p_{x;\alpha} \vec{e}_\theta + p_{y;\alpha} \vec{e}_\phi \quad (48)$$

are vectors in the tangential plane. Thus we can gauge them as

$$\vec{p}_\alpha \rightarrow \vec{p}_\alpha - q_\alpha \vec{A}. \quad (49)$$

Here, q_α is a LEC that can be adjusted to data by means of a single inter-band transition from the rotational band with the vibrational band head at an excitation energy $2\omega_\alpha$ to the ground-state band.

Why do the charges q_α depend on the mode χ_α ? The eigenmodes χ_α solve the Helmholtz equation (29). They differ from each other. The effective charge each mode carries is specific to that mode. As is the case for the energies ω_α , a microscopic calculation of that charge would require a model for the shape of the deformed nucleus and for the boundary conditions imposed at the surface.

4.3. From the EFT to effective theories

The dynamical variables in our EFT for deformed nuclei are the rotational degrees of freedom (θ, ϕ) and the amplitudes (x_α, y_α) of the NG modes. Both the excitation energies ω_α and the charges q_α are mode-specific parameters. The number of parameters increases with the number of modes considered. It is, therefore, tempting to confine attention to the (few) modes that are below the breakdown energy Λ of the EFT. That truncation reduces the EFT to an effective theory. In references [7, 8, 9], aspects of such an effective theory were investigated and used. We discuss the relation between that effective theory and the present EFT.

For simplicity we restrict the EFT developed in Sect. 3 to the lowest excitations with energy $2\omega_0$ as discussed in Subsection 4.1. At order $\mathcal{O}(\Omega)$ we deal with two two-dimensional isotropic harmonic oscillators. At order $\mathcal{O}(\xi)$ the oscillators are coupled to a rigid rotor. That yields two rotational bands with identical moments of inertia on top of both, the $k = 0$ band head and the degenerate $|k| = 2$ band head.

In contradistinction, the effective theory [7, 8, 9] starts from the emergent symmetry breaking of five quadrupole degrees of freedom d_μ . The components $d_{\pm 1}$ are replaced by two rotation angles (θ, ϕ) and become the modes with excitation energy ξ . The component d_0 and the components $d_{\pm 2}$ have the energy scale Ω . At order $\mathcal{O}(\Omega)$ the Hamiltonian is a three-dimensional axially symmetric harmonic oscillator with frequency $\tilde{\omega}_0$ for the d_0 degree of freedom and frequency $\tilde{\omega}_2$ for the $d_{\pm 2}$ degrees of freedom. The frequencies $\tilde{\omega}_0$ and $\tilde{\omega}_2$ have magnitude Ω but are not necessarily equal to each other. The lowest vibrational excitations have energy $\tilde{\omega}_0$ (one quantum in the d_0 mode) or energy $\tilde{\omega}_2$ (one angular excitation of the two-dimensional isotropic oscillator in the $d_{\pm 2}$ modes). At order ξ these oscillators are coupled to the rigid rotor with the degrees of freedom (θ, ϕ) . That yields a rotational band with $k = 0$ on top of the d_0 excitation and another one with $k = 2$ on top of the $d_{\pm 2}$ excitation. Thus, at order ξ the spectra of the effective theory and of the EFT differ from each other because the former has nondegenerate frequencies $\tilde{\omega}_0 \neq \tilde{\omega}_2$ while the latter has equal frequencies $2\omega_0$ for the $k = 0$ and $|k| = 2$ bands. In the effective theory, the difference in frequencies $\tilde{\omega}_0 - \tilde{\omega}_2$ is expected to be small compared to Ω and is viewed as a

higher-order correction. In that sense the effective theory and the EFT are equivalent low-energy theories.

The EFT for deformed nuclei yields a systematic approach that is based on symmetry principles alone. We mention two examples that show how within that approach, small deviations between data and the traditional collective models can be understood and addressed. These are the variation of the moment of inertia with the vibrational band head [8], and the weak inter-band $E2$ transitions [9]. We do so in the framework of the effective theory which, as just pointed out, is a simplified version of the EFT.

In nuclei, the moments of inertia (or rotational constants) of rotational bands on top of different vibrational excitations differ by relatively small amounts. Examples were given in Subsection 4.1. In the effective theory, rotational constants at order ξ have magnitude ξ^{-1} and for different rotational bands are equal. Taking account of higher-order corrections of relative order ξ/Ω , consistency with the data in $^{166,168}\text{Er}$ and ^{232}Th is attained [8].

For $E2$ transitions the effective theory predicts that intra-band transitions are strong and that inter-band transitions are suppressed by a factor of order ξ/Ω . Gauging of the effective theory shows that the inter-band transitions are governed by two additional LECs. These parameters do not appear in the traditional collective models, causing the latter to overpredict the faint inter-band transition strengths by factors 2–10 [28]. The effective theory remedies this problem [9] and thereby offers a solution to a long-standing discrepancy. Nuclei are non-rigid rotors. Within the EFT it becomes clear, for instance, that deviations from rigid-rotor expectations for the spectrum and quadrupole transitions are similar in relative size.

These examples show that a systematic and controlled approach to nuclear deformation is possible, and that small but significant deviations between data and collective models can be understood and addressed. Within the EFT such problems are not treated by simply adding terms with additional fit parameters. Instead, arguments of symmetry alone are used to determine which corrections arise at each order of power counting. The procedure is unambiguous. Consistency of the EFT approach requires that LECs are of natural size, i.e., have a magnitude that is in agreement with expectations from the power counting. The procedure also shows which additional assumptions are made when correction terms are used in the traditional collective models.

The expectation that LECs are of natural size leads to simple estimates of the theoretical uncertainties at any order of the power counting. For deformed nuclei, such uncertainty estimates for $B(E2)$ transitions suggest, for instance, that it might be profitable to remeasure or re-evaluate data for certain intra-band transitions [9]. Most interestingly, the EFT approach can also be used to truly quantify uncertainties. Assumptions about the natural size and distribution of LECs can be quantified as priors, and Bayesian statistics can be used to quantify theoretical uncertainties as degree-of-belief intervals that have a statistical meaning [55, 56, 57, 58, 59, 60].

5. Summary

We have reviewed the EFT approach to nuclei with intrinsically deformed but axially symmetric ground states. The deformation is viewed as a case of emergent symmetry breaking, the analogue of spontaneous symmetry breaking in infinite systems. Accordingly, our EFT extends well-known approaches to spontaneous

symmetry breaking in infinite systems to emergent symmetry breaking characteristic of finite systems. That is done using, in addition to the familiar Nambu-Goldstone modes, additional modes that account for nuclear rotation. The Hamiltonian consists of invariants that are constructed from the said modes using symmetry arguments alone. Each invariant is multiplied by a constant that has to be fitted to data. The invariants are ordered by power-counting arguments. In leading order, the Hamiltonian describes rotations and vibrations, each vibrational state serving as band head of a rotational band. Terms of higher order allow for a systematic improvement and lift degeneracies. That Hamiltonian governs the spectra of deformed nuclei at low excitation energies. The construction leaves no room for guess work: Each invariant and its order are well defined. These facts make it possible to address small but significant differences between data and traditional collective models. Examples are changes of the moments of inertia as the band head changes, or the magnitude of the faint inter-band $E2$ transitions.

The Nambu-Goldstone modes are defined in the coset space $SO(3)/SO(2)$. The resulting non-linear realization of symmetry breaking is at the heart of the EFT. Upon quantization the Nambu-Goldstone modes give rise to the nuclear vibrational modes. Very recently, low-energy vibrations in spherical nuclei have also been approached in an EFT [60]. Unlike the present approach, the EFT for such nuclear vibrations is based on the usual linear realization of rotational symmetry. That approach suggests that certain isotopes of Ni, Ru, Pd, Cd, and Te can be viewed as anharmonic quadrupole oscillators. The approach describes low-lying spectra and electromagnetic properties consistently within quantified theoretical uncertainties.

The EFT approach to heavy nuclei can be extended in several directions. Most interesting is probably the coupling of fermionic degrees of freedom to the bosonic fields discussed in this paper. That approach – similar in spirit to halo EFT – could open the way towards a model-independent theory of odd-mass and odd-odd heavy nuclei. Another possibility is an extension of our parity-conserving EFT to emergent parity breaking [61]. Octupole excitations, i.e., states with spin/parity $I^\pi = 1^-, 2^-, 3^-, \dots$ are low-lying vibrations in rare-earth nuclei and the lowest-lying vibrations in the actinides.

Ultimately, we wish to better understand how collective modes arise in complex nuclei, and to reliably quantify theoretical uncertainties. EFT approaches to heavy nuclei have the potential to deliver both.

Acknowledgments

One of us (HAW) acknowledges support by the Simons Center for Geometry and Physics at Stony Brook University where part of this paper was written. TP's work is supported in part by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, under award No. DE-FG02-96ER40963 (University of Tennessee), and under contract No. DEAC05-00OR22725 (Oak Ridge National Laboratory).

Appendix A. Transformation properties under rotations

In this Appendix, we derive the transformation properties of the degrees of freedom employed in the EFT. We also use Noether's theorem for the derivation of angular momentum as the conserved quantity, and show that the monopole gauge fields change under rotations by a total gradient.

A rotation $\delta\vec{\alpha} = (\delta\alpha_x, \delta\alpha_y, \delta\alpha_z)$ about infinitesimal angles $\delta\alpha_k$ around the laboratory $k = x, y, z$ axes changes the radial unit vector \vec{e}_r and the unit vectors \vec{e}_θ and \vec{e}_ϕ in the tangential plane at the point (θ, ϕ) as

$$\vec{e}_j(\theta, \phi) \rightarrow \vec{e}_j(\theta, \phi) + \delta\vec{\alpha} \times \vec{e}_j(\theta, \phi) . \quad (\text{A.1})$$

Here $j = r, \theta, \phi$. The rotation $\delta\vec{\alpha}$ also changes the point (θ, ϕ) on the sphere to $(\theta + \delta\theta, \phi + \delta\phi)$, and the three unit vectors at this rotated point are

$$\begin{aligned} \vec{e}_r(\theta + \delta\theta, \phi + \delta\phi) &= \vec{e}_r(\theta, \phi) + \delta\theta\vec{e}_\theta(\theta, \phi) + \delta\phi\sin\theta\vec{e}_\phi(\theta, \phi) , \\ \vec{e}_\theta(\theta + \delta\theta, \phi + \delta\phi) &= \vec{e}_\theta(\theta, \phi) - \delta\theta\vec{e}_r(\theta, \phi) + \delta\phi\cos\theta\vec{e}_\phi(\theta, \phi) , \\ \vec{e}_\phi(\theta + \delta\theta, \phi + \delta\phi) &= \vec{e}_\phi(\theta, \phi) - \delta\phi[\sin\theta\vec{e}_r(\theta, \phi) + \cos\theta\vec{e}_\theta(\theta, \phi)] . \end{aligned} \quad (\text{A.2})$$

Equating the expressions on the right-hand sides of Eq. (A.1) for $j = r$ and of the first of Eqs. (A.2) yields

$$\begin{pmatrix} \delta\theta \\ \delta\phi \end{pmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi\cot\theta & -\sin\phi\cot\theta & 1 \end{bmatrix} \begin{pmatrix} \delta\alpha_x \\ \delta\alpha_y \\ \delta\alpha_z \end{pmatrix} . \quad (\text{A.3})$$

That expression shows that the rotation $\delta\vec{\alpha}$ induces a non-linear transformation of the angles (θ, ϕ) . It follows that a scalar function $f(\theta, \phi)$ changes under the rotation $\delta\vec{\alpha}$ as

$$f(\theta - \delta\theta, \phi - \delta\phi) = f(\theta, \phi) - \delta\theta\partial_\theta f(\theta, \phi) - \delta\phi\partial_\phi f(\theta, \phi) \rightarrow f(\theta, \phi) , \quad (\text{A.4})$$

with $\delta\theta$ and $\delta\phi$ given by Eq. (A.3).

We consider a vector $\vec{a} = a_\theta\vec{e}_\theta(\theta, \phi) + a_\phi\vec{e}_\phi(\theta, \phi)$ in the tangential plane at the point (θ, ϕ) . Under the rotation $\delta\vec{\alpha}$, a radial vector is transformed into a radial vector and a tangential vector into a tangential vector. However, the rotated pair $\vec{e}_\theta, \vec{e}_\phi$ of tangential vectors does not coincide with the corresponding pair of tangential basis vectors at the point reached by the rotation. In general, the two pairs differ by a rotation in the tangential plane. We now determine the infinitesimal value of angle of that rotation.

Under a rotation $\delta\vec{\alpha}$, the basis vectors in the tangential plane transform as in Eq. (A.1) for $j = \theta, \phi$. It is straightforward to take scalar products of these vectors with the basis vectors $\vec{e}_\theta(\theta + \delta\theta, \phi + \delta\phi)$ and $\vec{e}_\phi(\theta + \delta\theta, \phi + \delta\phi)$ in the tangential plane at $(\theta + \delta\theta, \phi + \delta\phi)$ as defined in Eqs. (A.2). Alternatively we might use all three Eqs. (A.2), solve for the basis vectors at $(\theta + \delta\theta, \phi + \delta\phi)$, re-express the rotated tangent vectors (A.1) in terms of the latter, and compute the scalar products. In either case we obtain the transformation law for the basis vectors in the tangential plane

$$\begin{pmatrix} \vec{e}_\theta \\ \vec{e}_\phi \end{pmatrix} \rightarrow \begin{bmatrix} 1 & \delta\gamma \\ -\delta\gamma & 1 \end{bmatrix} \begin{pmatrix} \vec{e}_\theta \\ \vec{e}_\phi \end{pmatrix} . \quad (\text{A.5})$$

Here

$$\delta\gamma = \frac{\cos\phi}{\sin\theta}\delta\alpha_x + \frac{\sin\phi}{\sin\theta}\delta\alpha_y . \quad (\text{A.6})$$

For the components of the vector $\vec{a} = a_\theta\vec{e}_\theta(\theta, \phi) + a_\phi\vec{e}_\phi(\theta, \phi)$ we obtain correspondingly

$$\begin{pmatrix} a_\theta \\ a_\phi \end{pmatrix} \rightarrow \begin{bmatrix} 1 & -\delta\gamma \\ \delta\gamma & 1 \end{bmatrix} \begin{pmatrix} a_\theta \\ a_\phi \end{pmatrix} . \quad (\text{A.7})$$

A rotation $\delta\vec{\alpha}$ induces a rotation of vectors in the tangential plane by the angle $\delta\gamma$. That is an alternative explanation of why the NG fields (which “live” in the

tangential plane) transform linearly under rotations albeit with a complex angle. Using Eqs. (A.7) and (A.6) yields the transformation properties of the NG fields

$$\begin{pmatrix} \delta\psi_x \\ \delta\psi_y \end{pmatrix} = \begin{bmatrix} -\psi_y \frac{\cos\phi}{\sin\theta} & -\psi_y \frac{\sin\phi}{\sin\theta} & 0 \\ \psi_x \frac{\cos\phi}{\sin\theta} & \psi_x \frac{\sin\phi}{\sin\theta} & 0 \end{bmatrix} \begin{pmatrix} \delta\alpha_x \\ \delta\alpha_y \\ \delta\alpha_z \end{pmatrix}. \quad (\text{A.8})$$

Employing the matrix elements in Eqs. (A.3) and (A.8) together with Noether's theorem yields the angular-momentum components

$$\begin{aligned} I_x &= -p_\theta \sin\phi - p_\phi \cos\phi \cot\theta + \frac{\cos\phi}{\sin\theta} \int d^3x (\psi_x p_y - \psi_y p_x), \\ I_y &= p_\theta \cos\phi - p_\phi \sin\phi \cot\theta + \frac{\sin\phi}{\sin\theta} \int d^3x (\psi_x p_y - \psi_y p_x), \\ I_z &= p_\phi \end{aligned} \quad (\text{A.9})$$

as the conserved quantities. This is Eq. (15).

We apply these results to the vector potential (21). Under the rotation $\delta\vec{\alpha}$ we have

$$\begin{aligned} \frac{\cot\theta}{r} \vec{e}_\phi &\rightarrow \frac{\cot\theta}{r} \vec{e}_\phi - \delta\gamma \frac{\cot\theta}{r} \vec{e}_\theta + \frac{\delta\theta}{r \sin^2\theta} \vec{e}_\phi \\ &= \frac{\cot\theta}{r} \vec{e}_\phi + \vec{\nabla}(\delta\gamma). \end{aligned} \quad (\text{A.10})$$

The rotation changes the monopole vector potential by the gradient of a scalar function, i.e., by an unobservable gauge transformation. Hence, any combination

$$(\vec{p} - C \cot\theta \vec{e}_\phi)^2 \quad (\text{A.11})$$

with a rotational scalar C is invariant under rotations. That explains the occurrence of terms involving $\cot^2\theta$ in the higher-order corrections of the EFT.

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